analogy, are found to be in agreement with the experiment. Although the present technique requires experimental data, in the form of the vortex core locations, the model does account for the previously ignored mass entrainment of the vortex core.

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Determination of Bending Influence Coefficients from Bending Slope Data

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BENDING influence coefficients for a cantilever beam are defined as follows: 1) $H_h(x,\bar{x})$ = vertical deflection at "x" due to unit force at "$\bar{x}$" (in./lb); 2) $H_\theta(x,\bar{x})$ = vertical deflection at "x" due to unit moment at "$\bar{x}$" (in./in. lb); 3) $\theta_h(x,\bar{x})$ = angular deflection at "x" due to unit force at "$\bar{x}$" (rad/lb); 4) $\theta_\theta(x,\bar{x})$ = angular deflection at "x" due to unit moment at "$\bar{x}$" (rad/in. lb). Direct determination of these quantities would normally require several sets of measurements for each separate loading. It will be shown that all of these quantities can be found either directly or by simple quadrature from a single set of measurements, namely, the angular deflection distribution due to a unit moment applied at the tip.

The analytical expressions for the various influence coefficients in terms of bending stiffness, EI, are

$$\begin{cases} H_h(x,\bar{x}) = \displaystyle\int_0^x \frac{(x-\xi)(\bar{x}-\xi)}{EI(\xi)}\,d\xi, & x \leqq \bar{x} \\ H_h(x,\bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(x-\xi)(\bar{x}-\xi)}{EI(\xi)}\,d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} H_\theta(x,\bar{x}) = \displaystyle\int_0^x \frac{(x-\xi)}{EI(\xi)}\,d\xi, & x \leqq \bar{x} \\ H_\theta(x,\bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(x-\xi)}{EI(\xi)}\,d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_h(x,\bar{x}) = \displaystyle\int_0^x \frac{(\bar{x}-\xi)}{EI(\xi)}\,d\xi, & x \leqq \bar{x} \\ \theta_h(x,\bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(\bar{x}-\xi)}{EI(\xi)}\,d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_\theta(x,\bar{x}) = \displaystyle\int_0^x \frac{d\xi}{EI(\xi)}, & x \leqq \bar{x} \\ \theta_\theta(x,\bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{d\xi}{EI(\xi)}, & x \geqq \bar{x} \end{cases}$$

From these equations it is seen that

$$H_h(x,\bar{x}) = H_h(\bar{x},x)$$
$$H_\theta(x,\bar{x}) = \theta_h(\bar{x},x)$$
$$\theta_\theta(x,\bar{x}) = \theta_\theta(\bar{x},x)$$

If $\phi(x)$ is the angular deflection at x due to a unit moment at the tip, then

$$\phi(x) = \int_0^x \frac{d\xi}{EI(\xi)}$$

Hence

$$\frac{1}{EI(x)} = \frac{d\phi}{dx}$$

Inserting the expression for $(1/EI)$ in the equations for the influence coefficients and simplifying, we obtain

$$\begin{cases} H_h(x,\bar{x}) = (x+\bar{x})\displaystyle\int_0^x \phi(\xi)\,d\xi - 2\int_0^x \phi(\xi)\xi\,d\xi & x \leqq \bar{x} \\ H_h(x,\bar{x}) = (x+\bar{x})\displaystyle\int_0^{\bar{x}} \phi(\xi)\,d\xi - 2\int_0^{\bar{x}} \phi(\xi)\xi\,d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} H_\theta(x,\bar{x}) = \displaystyle\int_0^x \phi(\xi)\,d\xi & x \leqq \bar{x} \\ H_\theta(x,\bar{x}) = (x-\bar{x})\phi(\bar{x}) + \displaystyle\int_0^{\bar{x}} \phi(\xi)\,d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_h(x,\bar{x}) = (\bar{x}-x)\phi(x) + \displaystyle\int_0^x \phi(\xi)\,d\xi & x \leqq \bar{x} \\ \theta_h(x,\bar{x}) = \displaystyle\int_0^{\bar{x}} \phi(\xi)\,d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_\theta(x,\bar{x}) = \phi(x) & x \leqq \bar{x} \\ \theta_\theta(x,\bar{x}) = \phi(\bar{x}) & x \geqq \bar{x} \end{cases}$$

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Stiffness of Orthotropic Materials and Laminated Fiber-Reinforced Composites

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Introduction

CHU[1] considered a laminated fiber-reinforced composite with 40% of the fibers aligned with the 1-axis in Fig. 1 and the remaining 60% at ±45° to the 1-axis. His description of the laminate unfortunately did not include the number of layers and the stacking sequence. He was perplexed by the observation that the Young's modulus in an x-direction (E_x) other than the 1-direction was larger than in the 1-direction (E_1).† He concluded

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† See Ref. 2 for notation.

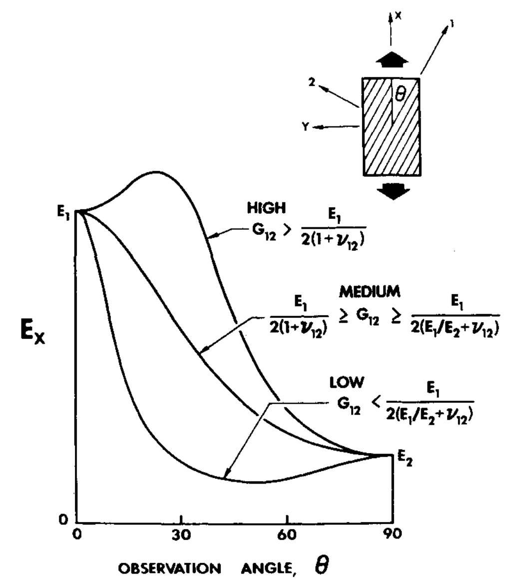

Fig. 1 Schematic variation of E_x with θ for various shear moduli.

his laminate was macroscopically orthotropic and proceeded to derive some stiffness relations for orthotropic materials in order to explain his observation.

The purpose of this Note is to clarify, correct, and extend Chu's results. First, Chu's laminate is analyzed. Then, the stiffness characteristics of orthotropic materials are discussed. Finally, some concluding observations are made on orthotropic materials and, more generally, on laminated composite materials.

Laminate Analysis

First of all, Chu should not have used material properties E_x and E_1, etc., to characterize a laminate. Instead, he should have used classical laminate stiffnesses,[2] A_{ij}, B_{ij}, and D_{ij}, which involve the lamination characteristics of thickness and orientation of each layer in addition to the material properties. We will proceed with a classical lamination theory analysis.

First, we will presume Chu's laminate is symmetric about the middle surface, so as to avoid coupling between bending and extension which, if present, would invalidate the applicability of Chu's derivation. Second, we will presume that each layer of the laminate has the same thickness and same material (a probable situation since Chu's organization uses primarily equal thickness layers of the same material in boron/epoxy and graphite/epoxy aircraft components). Because of the equal thickness layer requirement and the original (incomplete) stipulation that 40% of the fibers are at 0° and the remaining 60% at ±45°, the laminate must have ten layers or some multiple thereof. We shall further assume that 30% of the layers are at +45° and 30% of the layers are at −45° to the reference axis. Accordingly, such a laminate could have the stacking sequence [+45/−45/+45/0/0/0/0/−45/+45/−45]. However, this laminate is obviously not symmetric about the middle surface so it does not satisfy the first assumption. The simplest laminate that satisfies all assumptions and requirements has 20 layers and could have the stacking sequence $[+45/-45/+45/-45/+45/-45/0/0/0/0/]_{\text{symmetric}}$. In fact, any stacking sequence with 8 layers at 0°, 6 at +45°, and 6 at −45° arranged symmetrically about the middle surface will satisfy the requirements.

The laminate stiffnesses, A_{16}, A_{26}, D_{16}, and D_{26}, are crucial to Chu's contention that his laminate is macroscopically orthotropic. First,

$$A_{16} = \sum_{k=1}^{20} (\bar{Q}_{16})_k (z_k - z_{k-1}) \tag{1}$$

But, $(\bar{Q}_{16})_k = 0$ for all 0° layers, $\bar{Q}_{16}(+\theta) = -\bar{Q}_{16}(-\theta)$ for all 45° layers, and $(z_k - z_{k-1}) = t_k = t/20$ where t is the laminate thickness. Thus, $A_{16} = 0$ and, similarly, $A_{26} = 0$. However, from the definition of D_{16}

$$D_{16} = 1/3 \sum_{k=1}^{20} (\bar{Q}_{16})_k (z_k{}^3 - z_{k-1}{}^3) \tag{2}$$

it is easily shown that D_{16} is not zero. Similarly, D_{26} is not zero.

Accordingly, the force-strain-moment-curvature relations for Chu's laminate are of the form

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} \tag{3}$$

Such a laminate behaves like an orthotropic material when subjected to in-plane loads because $A_{16} = A_{26} = 0$. However, when subjected to bending moments, Chu's laminate has coupling between normal moments and twisting curvature as well as between twisting moment and normal curvatures. Thus, his description macroscopically orthotropic is incorrect. His laminate is macroscopically orthotropic under extension, but under bending it is macroscopically anisotropic. Thus, his laminate is macroscopically anisotropic when the loading is not specified (as he did not). The terminology must be independent of loading or include specification of the loading. The label macroscopically orthotropic is obviously too simple for use in describing laminate behavior. Instead, the laminate stiffness matrix should always be used to characterize laminate behavior.

With the foregoing comments in mind, we can proceed to describe Chu's laminate, with reservations, as an extensionally orthotropic laminate and hence as an orthotropic material when only extensional loads are applied. Accordingly, we can observe the extensional stiffness A_{11} as a function of rotation from the principal material directions or equivalently (but not equally in magnitude[3]) the Young's modulus E_x as a function of rotation.

Stiffness Characteristics of Orthotropic Materials

Chu's observations on the behavior of orthotropic materials are generally correct. For a single orthotropic layer (presumably of a laminate, but such a stipulation is not essential), it may have surprised Chu that the shearing modulus G_{12} can cause either a higher E_x than E_1 or lower E_x than E_2. However, these facts have been observed for many years implicitly even if not explicitly. The specific ranges of G_{12} for which various stiffness behaviors occur have not been determined before to the author's knowledge. Those ranges are the real contribution of Chu's Note. The following discussion is a review, correction, and more complete explanation of Chu's results which are of significant interest to the composite materials community.

Chu stated that a) E_x is greater than both E_1 and E_2 for some values of θ in Fig. 1 if

$$G_{12} > E_1/[2(1+\nu_{12})] \tag{4}$$

and b) E_x is less than both E_1 and E_2 for some values of θ in Fig. 1 if

$$G_{12} < E_1/[2(E_1/E_2+\nu_{12})] \tag{5}$$

Although these results are correct, it is worth noting that to prove them requires investigation of the character of the stationary values of

$$E_1/E_x = \cos^4\theta + (E_1/E_2)\sin^4\theta + \tfrac{1}{4}[(E_1/G_{12}) - 2\nu_{12}]\sin^2 2\theta \tag{6}$$

which entails examination of the sign of up through the fourth derivative of Eq. (6). Although in principal the technique is straightforward, the differential calculus operations take twelve pages.

Incidentally, Chu's Eq. (6) is incorrect; it would be both correct and follow from his Eq. (4) if the bottom line of his Eq. (6) were

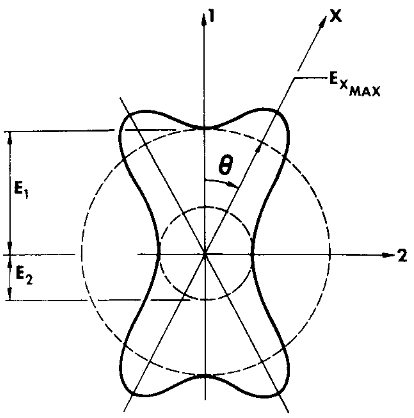

Fig. 2 Variation of Young's modulus, E_x, with rotation from principal material directions for an orthotropic material with high G_{12}.

$$\gamma_{xy}/2 = -(m_x/2E_1)\sigma_x - (m_y/2E_1)\sigma_y + (1/2G_{xy})\tau_{xy} \qquad (7)$$

Also, Chu's Eq. (11) should read

$$\frac{d(E_1/E_x)}{d\theta} = -4(1-a-4b)\cos^3\theta\sin\theta - 2(4b-2a)\sin\theta\cos\theta = 0 \qquad (8)$$

The foregoing results can be summarized by the observation that an orthotropic material can have an apparent Young's modulus in nonprincipal material directions that either exceeds or is less than the Young's moduli in both principal material directions. The graphical interpretation of Eqs. (4) and (5) with accompanying restrictions is shown in Fig. 1. The "intuitively predicted" variation of E_x with θ is given by the curve labeled

$$E_1/[2(1+\nu_{12})] \geqq G_{12} \geqq E_1/[2(E_1/E_2+\nu_{12})] \qquad (9)$$

The visual impact of Fig. 1 is unavoidable; this graphical display is particularly effective in illustrating the three types of behavior ["high," "medium," and "low" G_{12} for Eqs. (4, 9, and 5), respectively].

Physically, the results mean that the shearing modulus of an orthotropic material has a strong influence on the character of the Young's moduli in nonprincipal material directions. If G_{12} is low as in woven materials, then E_x at say $\theta = 45°$ will be lower than E_2. For example, pull on an ordinary pillow case or bedsheet in the two orthogonal fiber directions and then at 45° to the fibers. In the latter case, E_x is sensibly much lower than either E_1 or E_2. A structurally more practical but less vivid example of a "low" G_{12} material is boron/epoxy with material properties.

$$E_1 = 30\times10^6 \text{ psi} \quad E_2 = 3\times10^6 \text{ psi} \quad G_{12} = 1\times10^6 \text{ psi}$$
$$\nu_{12} = 0.3$$

Obviously, Eq. (5) is satisfied, but the inequality is not strong so the minimum value of E_x is only slightly less than E_2. Examples of the intermediate behavior ("medium" G_{12}) characterized by Eq. (9) include glass/epoxy and high modulus graphite/epoxy although their shear moduli are nearly low enough to qualify them as "low" G_{12} materials. Apparently, Chu's extensionally orthotropic laminate has a "high" value of G_{12} such that Eq. (4) is satisfied although he did not give any material properties. Generally, "high" G_{12} materials will not occur naturally and will not likely occur with woven materials because the matrix material is usually quite flexible and hence has a low G itself. Thus, the composite has a low G_{12} since the matrix shear modulus is the dominant influence on the composite shear modulus. Apparently, one of the few practical ways of achieving "high" G_{12} materials is to increase G_{12} for a composite not by increasing G of the matrix material but by adding angle-ply layers to increase A_{66}, the laminate shearing stiffness. However, this possibility often involves more than just an orthotropic material; care must be taken to perform an adequate laminate analysis.

Note that orthotropic materials with either high or low G_{12} do not have their largest and smallest Young's moduli in orthogonal directions as do orthotropic materials with medium G_{12}. As an example, consider E_x for an orthotropic material with high G_{12} shown in Fig. 2 for a complete revolution in observation angle. There, the largest E_x values occur at roughly $\theta = \pm 30°$ and $\theta = \pm 150°$ with the smallest E_x values at $\theta = \pm 90°$. Thus, not only are there nonorthogonal directions for maximum and minimum values, but the maximum occurs at *four* values of θ instead of only two as for materials with medium G_{12}. The significance of principal material axes is that they are axes of *material symmetry* but do not necessarily coincide with maximum or minimum values of the material properties.

Conclusions

Laminated composite materials must be completely specified as to stacking sequence, number and orientation of layers, and lamina material properties because of the complex nature of the laminate stiffness A_{ij}, B_{ij}, and D_{ij}. Care must be taken to use proper terminology resulting from a laminate analysis to avoid misleading nomenclature.

Orthotropic materials, at arbitrary observation angles, can have larger or smaller values of the Young's modulus E_x than E_1 or E_2, the Young's moduli in principal material directions. The values of E_x depend on the value of the shearing modulus in principal material coordinates, G_{12}. Practical examples of materials with low, medium, and high G_{12} are given.

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Char Formation in Ablatives

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CHARRING ablators have been employed as a heat shield during re-entry of space capsules[1] and for protection of rocket nozzles. Such ablators usually decompose to form porous carbon (char) and low molecular weight gases. Heat protection of the space vehicle is provided[2] by a) conductive heat transfer through the char and convective heat transfer through the entrapped volatile products, b) transpiration, c) endothermic chemical reaction of decomposition products, d) reradiation from the char front surface, and e) thickening of the boundary layer. In order to understand the nature of ablating boundary layer, detailed measurements under turbulent ablating conditions have been made recently[3] and the effect on surface heat transfer of gas phase chemical reactions involving ablation products have also been studied.[4] Many new types of ablating materials have been reported during recent years. However, it seems that the chemistry of char formation has not received the attention which it deserves. In this communication, we shall discuss the chemistry of char formation based on the nature of decomposition products and the bond energy considerations.

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